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Revisiting the moving force problem

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Abstract

The problem of the vibration of a beam subject to a travelling force is considered. The purpose of the study is to develop simple tools for finding the maximum deflection of a beam for any given velocity of the travelling force. It is shown that, for given boundary conditions, there exists a unique response–velocity dependence function. A technique to determine this function is suggested, which is based on the assumption that the maximum beam response can be adequately approximated by means of the first beam mode. To illustrate this, the maximum response function is calculated analytically for a simply supported (SS) beam and constructed numerically for a clamped–clamped beam. The effect of the higher modes on the maximum response is investigated, and the relative error of the one-mode approximation for a SS beam is constructed. The estimates obtained substantiate the assumption about adequacy of the one-mode approximation in a wide range of velocities; in particular, the relative error in the neighborhood of the velocity that results in the largest response is less than one percent.

1. Introduction

The problem of calculation of the response of a beam subject to a travelling force, the moving force problem (MFP), is a relatively simple one compared to those related to more realistic models of vehicle–bridge interaction such as the moving mass and moving oscillator models. Many methods for solving this problem were developed and discussed in the literature (see, e.g., Refs. [1,2]; and references therein). One of the most well-known techniques is to represent the solution

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in a series form in terms of the eigenfunctions of the beam [2–5]. If the travelling force is constant, the time-dependent coefficients can be obtained in an analytical form [2–4], which makes it easy to find a solution to any particular problem. However, in many cases, it is not just the response of the beam (which is a function of the spatial variable and time and depends on several parameters such as beam characteristics, the magnitude of the force and its velocity) that the designer is interested in. Given a particular beam and the magnitude of the travelling force, the following characteristics may be of interest: the maximum response of the beam for a given velocity of the force, the maximum response over all possible velocities and the velocity at which it occurs, the dependence of the forced response and the amplitude of the free vibration on the velocity, and the like. The numerical determination of such characteristics requires solving the MFP repeatedly and the result is not generally applicable to other beams with different inertia and stiffness distributions.

The purpose of this study is to examine the dependence of the maximum response of a beam on the velocity of the travelling force and to provide the engineer who faces the MFP with some *simple tools* that make it possible to get some desired characteristics immediately by using routine arithmetic operations. The major contribution of this work is the determination of unique amplitude-velocity dependence functions for simply supported (SS) and clamped–clamped (CC) beams, which allow for convenient assessment of the maximum beam response without intensive computations. It is shown that the maximum response can adequately be approximated by one term of the expansion. The effect of the higher terms of the expansion is investigated, and estimates of relative error of the one-mode approximation are given, which substantiate the high accuracy of the one-mode approximation. The maximum response of the unit dimensionless beam with given boundary conditions (BCs) is either calculated analytically or computed numerically only once. The application of the results obtained to a particular beam is straightforward and requires only rescaling the parameters.

Although the moving force model is a relatively simple one, it is worthy of examination for the following reasons. First, the knowledge of certain integral characteristics (rather than particular solutions) of the moving force solution, which are obtained without any additional effort, can be very useful for engineers and researchers when examining more realistic models of moving vehicles, providing them with reasonable approximations of the desired results and helping to devise a plan of attack on the problem. Second, the analysis of this model is still not trivial, and examples of misinterpretation of the solution can be found in some textbooks (see, e.g., discussion in Section 4.1).

2. Problem statement

The equation governing vibration of a uniform beam subject to a constant force F traversing the beam with a constant velocity v is given by

$$\rho \frac{\partial^2}{\partial t^2} w + EI \frac{\partial^4}{\partial x^4} w = F\delta(x - vt), \quad 0 \leqslant x \leqslant L, \ 0 \leqslant t \leqslant L/v, \tag{1}$$

subject to given BCs. Introducing the dimensionless notation

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \Omega t, \quad \Omega = \frac{1}{L^2}\sqrt{\frac{EI}{\rho}}, \quad \bar{w} = \frac{EIw}{FL^3}, \quad \bar{v} = \frac{v}{L\Omega}, \quad \bar{\delta}(\cdot) = L\delta(\cdot),$$

we can write Eq. (1) in the form

$$\frac{\partial^2}{\partial \bar{t}^2}\bar{w} + \frac{\partial^4}{\partial \bar{x}^4}\bar{w} = \bar{\delta}(\bar{x} - \bar{v}\bar{t}), \quad 0 \leqslant \bar{x} \leqslant 1, \ \ 0 \leqslant \bar{t} \leqslant 1/\bar{v}. \tag{2}$$

Then it follows that the response $\bar{w}$ is a function of only three parameters: $\bar{x}, \bar{t}$, and $\bar{v}$.

Let us set a problem of finding the function $\bar{W}(\bar{v}) = \max_{\bar{x}} \max_{\bar{t}} |\bar{w}(\bar{x}, \bar{t}; \bar{v})|$, which shows the maximum deflection of the beam for a given velocity $\bar{v}$. As can be easily seen, this function depends only on the dimensionless velocity and, thus, is unique for any beam with the same BCs. Given the parameters ρ and EI, the velocity v, and the magnitude F of the travelling force, the maximum response of any beam with the same BCs is

$$W(v) = \max_x \max_t |w(x, t; v)| = \frac{FL^3}{EI} \bar{W}\left(vL\sqrt{\frac{\rho}{EI}}\right). \tag{3}$$

In what follows, we will use dimensionless quantities and will drop the overbars for simplicity of notation when this does not result in any confusion.

3. Approach

The main difficulty associated with finding the function $W(v)$ is that, generally, this involves solving problem (2) numerically many times to calculate the response $w(x, t; v)$ for different values of v. Moreover, the global maximum with respect to time can occur at the moment when the force leaves the beam, which requires solving the problem of free vibration. Numerical finding of $\max_x \max_t |w(x, t; v)|$ is also a rather complicated problem, since, for a given v, $|w(x, t; v)|$ is a non-smooth function of two variables and may have many local maxima (this is further illustrated by the example of a SS beam).

The approach we suggest in this work to find the function $W(v)$ is based on an approximate solution to problem (2) and is as follows. The solution to the problem can be represented in the series form

$$w(x, t) = \sum_{n=1}^{\infty} \varphi_n(x) q_n(t), \tag{4}$$

where

$$q_n(t) = \int_0^t \frac{\varphi_n(v\tau)}{\omega_n} \sin \omega_n(t - \tau) \, \mathrm{d}\tau, \tag{5}$$

and ω_n and $\varphi_n(x)$ are eigenfrequencies and eigenfunctions of the beam respectively.

It is well-known (e.g. Ref. [2]) that the contribution of the first mode in the response considerably exceeds the contribution of the others. This allows us to assume that the use of the first-mode approximation for the problem under consideration can be quite sufficient for our purposes. Mathematically, we assume that, when calculating $\max_x \max_t |w(x, t)|$, we can substitute the function

$$w_1(x, t) = q_1(t)\varphi_1(x), \tag{6}$$

for the exact solution $w(x, t)$ even if $w_1(x, t)$ poorly approximates $w(x, t)$ for certain x and t. It is almost evident that the relative error of calculation due to discarding the higher-order terms is minimal for those values of x and t, for which the response is maximal. The validity of this assumption will be further justified by the analysis of the contribution of the higher modes to the maximum response of an SS beam given in Section 4.3 and by results of numerical experiments.

The use of the approach outlined above results in a considerable simplification of the optimization problem. Indeed, instead of the function of two variables, we now deal with the function of one variable, $q_1(t)$, since the dependence on x is trivial: the maximum is attained at $x = 0.5$.

Given that the one-mode approximation to the maximum response is found, one can take other modes into account considering the contributions of those modes to the maximum response as perturbations of the one-mode solution. This technique is demonstrated in Section 4 by calculating the error of the one-mode approximation of the maximum response of a SS beam due to discarding the second and third modes (in practice, the effect of the higher modes can certainly be neglected). The function obtained is very useful: on the one hand, it gives an idea of relative error of the one-mode approximation; on the other hand, if the researcher is not satisfied with the accuracy of the one-mode approximation obtained, this function can be used to improve the solution.

For a SS beam, the coefficients $q_n(t)$ can be obtained in an explicit analytical form; the global maximum of $|q_1(t)|$ can also be found analytically, which results in an explicit form of the function describing the dependence of the maximum response on the dimensionless velocity. The analysis of this case is given in the next section.

For other BCs, the eigenfunctions of the beam have more complicated form, and the analytical examination becomes too involved. However, the numerical calculation of the time-dependent coefficients, as well as finding their maxima numerically, presents no difficulties. Plotting the results obtained versus dimensionless velocity, we get the desired function. In Section 5, we demonstrate this by constructing the function $W(v)$ for a CC beam.

4. SS Beam

4.1. Background results

The eigenfunctions and eigenfrequencies of the unit SS beam are well-known to be $\varphi_n(x) = \sqrt{2} \sin \lambda_n x$, $\omega_n = \lambda_n^2$, $\lambda_n = n\pi$, $n = 1, 2, \ldots$. Substituting the equation for $\varphi_n(x)$ into Eq. (5) and taking the integral by parts, we can write Eq. (4) in the form [3,4]

$$w(x, t) = \sum_{n=1}^{\infty} q_n(t)\sqrt{2} \sin \lambda_n x, \quad 0 \leqslant t \leqslant \frac{1}{v}, \tag{7}$$

where

$$q_n(t) = \frac{\sqrt{2}}{\omega_n(\alpha_n^2 - \omega_n^2)}(\alpha_n \sin \omega_n t - \omega_n \sin \alpha_n t), \quad \alpha_n = \lambda_n v. \tag{8}$$

In Ref. [4, p. 471], it is stated that the displacement becomes infinitely large when $\alpha_n = \omega_n$, i.e., when the force velocity is equal to $v_n = \omega_n/\lambda_n = n\pi$. This is, of course, not true. Although, when

$\alpha_n = \omega_n$, the denominator on the right-hand of Eq. (8) is zero, the numerator is zero as well. Applying L'Hospital's rule to the right-hand of Eq. (8) as $\alpha_n \to \omega_n$, we get

$$\lim_{\alpha_n \to \omega_n} q_n(t) = \lim_{\alpha_n \to \omega_n} \frac{\sqrt{2}(\sin \omega_n t - t\omega_n \cos \alpha_n t)}{2\omega_n \alpha_n} = \frac{\sin \omega_n t - t\omega_n \cos \omega_n t}{\sqrt{2}\omega_n^2},$$

which clearly shows that the $q_n(t)$ depends continuously on the velocity and is finite for any $t \leqslant 1/v$.

Similarly, in Ref. [3, p. 294], it is stated that Eq. (8) is valid only for $\alpha_n \neq \omega_n$ and that, otherwise, "a situation resembling resonance develops" (the solution infinitely grows in time). Clearly, this is not correct: no resonance can occur since the force traverses the beam for a finite time (only half a period of the first eigenvibration for $n = 1$), and a finite solution does exist. Moreover, as will be shown, the maximum of the beam response never occurs at these values of the velocity.

4.2. One-mode approximation to the solution

As discussed in Section 3, when looking for the maximum response $W(v)$, we use function (6) rather than $w(x, t)$. Depending on the velocity v, the maximum response may occur either when the force traverses the beam (forced vibration) or when it leaves it (i.e., during the free vibration of the beam). Clearly, in both cases, for any t, the maximum response with respect to x occurs at $x = 0.5$, and we need to find the maximum with respect to t. First, we consider the forced vibration.

4.2.1. Forced vibration
Let v be fixed, $v \neq v_1$, where $v_1 = \omega_1/\lambda_1 \equiv \pi$, and let us find the value of t for which $|q_1(t)|$ takes its maximum value in the interval $[0, T_p]$, where $T_p = 1/v$. To find a maximum, we differentiate Eq. (8)

$$\dot{q}_1(t) = \frac{\sqrt{2}\alpha_1}{\alpha_1^2 - \omega_1^2}(\cos \omega_1 t - \cos \alpha_1 t).$$

Since $v \neq v_1$, the right-hand side of this equation can be zero only if

$$\omega_1 t + \alpha_1 t = 2k\pi, \quad k = 1, 2, \ldots ,$$

or

$$\omega_1 t - \alpha_1 t = 2k\pi, \quad k = 1, 2, \ldots ,$$

the latter points corresponding to local minima of $q_1(t)$ or local maxima of $-q_1(t)$. Then, it follows that local maxima of $|q_1(t)|$ can take place at

$$t(\beta_1, k) = \frac{2k\pi}{\omega_1 + \alpha_1} \equiv \frac{2k\pi}{\omega_1(1 + \beta_1)}, \quad k = 1, 2, \ldots , \tag{9}$$

where β_1 is the dimensionless velocity, introduced for convenience of notation as

$$\beta_1 = \frac{\alpha_1}{\omega_1} \equiv \frac{v}{v_1} \equiv \frac{v}{\pi}.$$

Since we consider forced vibration, $t(\beta_1, k)$ must satisfy the inequality $t(\beta_1, k) \leqslant T_p$. By using this, it is not difficult to show that, for $1/(4*n + 1) < \beta_1 \leqslant 1/(4*n - 3)$, $|q_1(t)|$ has n local maxima and the

global maximum is attained at $t = t(\beta_1, n)$ and is given by

$$\max|q_1(t)| = \frac{\sqrt{2}}{\omega_1^2} \Phi_{1n}(\beta_1),$$

where

$$\Phi_{1n}(\beta_1) = \frac{1}{1 - \beta_1} \left| \sin \frac{2n\pi}{1 + \beta_1} \right|. \tag{10}$$

If $\beta_1 > 1$ ($v > v_1$), then $t(\beta_1, k) > T_p$ for all k. This implies that there are no maxima of $|q_1(t)|$ in the interval $[0, T_p)$, and the maximum response of the forced vibration takes place when the force is at the right end of the beam. The corresponding value of the time-dependent coefficient is obtained by substituting $t = T_p$ into Eq. (8) and taking into account that $\alpha_1 T_p = \pi$, giving

$$q_1(T_p) = \frac{\sqrt{2}\alpha_1}{\omega_1(\alpha_1^2 - \omega_1^2)} \sin \omega_1 T_p = \frac{\sqrt{2}\beta_1}{\omega_1^2(\beta_1^2 - 1)} \sin \frac{\pi}{\beta_1} \equiv \frac{\sqrt{2}}{\omega_1^2} \Phi_{10}(\beta_1), \quad \beta_1 \geqslant 1.$$

Let k vary from 1 to ∞, we denote by $\Phi_1(\beta_1)$ the function defined by the rule

$$\Phi_1(\beta_1) \equiv \Phi_{10}(\beta_1), \quad \text{for } \beta_1 \geqslant 1, \quad \Phi_1(\beta_1) \equiv \Phi_{1k}(\beta_1), \quad \text{for } \frac{1}{4k + 1} \leqslant \beta_1 \leqslant \frac{1}{4k - 3}. \tag{11}$$

In the beginning of this section, we assumed that $v \neq v_1$. However, as can be seen, the limits of the functions $\Phi_{10}(\beta_1)$ and $\Phi_{11}(\beta_1)$ as $\beta_1 \to 1$ exist and equal each other, and the function $\Phi_1(\beta_1)$ is thus defined for all values of β_1.

It can be checked directly that, for $\beta_1 \leqslant 1$,

$$\max_x \max_t |w_1(x, t)| = \sqrt{2} \max_{0 \leqslant t \leqslant T_p} |q_1(t)| = \frac{2}{\omega_1^2} \Phi_1(\beta_1) \equiv \frac{2}{\pi^4} \Phi_1(\beta_1),$$

i.e., the dependence of the maximum forced response on the velocity is given by the function $\Phi_1(\beta_1)$. The plot of the function $\Phi_1(\beta_1)$ is depicted in Fig. 1 by the solid line.

4.2.2. Free vibration

As was mentioned, the maximum response may take place after the force leaves the beam. The equation governing the free vibration of the beam is derived in Appendix A. Since we consider the one-mode approximation to the solution, the maximum response is assumed to be equal to the amplitude $|C_1(v)|$ of the first eigenvibration. Introduce the notation

$$\Phi_1^{free}(\beta_1) = \frac{2\beta_1}{|\beta_1^2 - 1|} \left| \sin \omega_1 \left(\frac{1}{2\pi} + \frac{T_p}{2} \right) \right| \equiv \frac{2\beta_1}{|\beta_1^2 - 1|} \left| \sin \frac{\pi}{2} \left(\frac{\beta_1 + 1}{\beta_1} \right) \right|. \tag{12}$$

Then, the amplitude is given by

$$|C_1(\beta_1)| = \frac{2}{\omega_1^2} \Phi_1^{free}(\beta_1). \tag{13}$$

The function $\Phi_1^{free}(\beta_1)$, depicted in Fig. 1 by the dashed line, is not specific to any particular SS beam and shows the dependence of the amplitude of the first beam eigenvibration on the dimensionless velocity of the moving force.
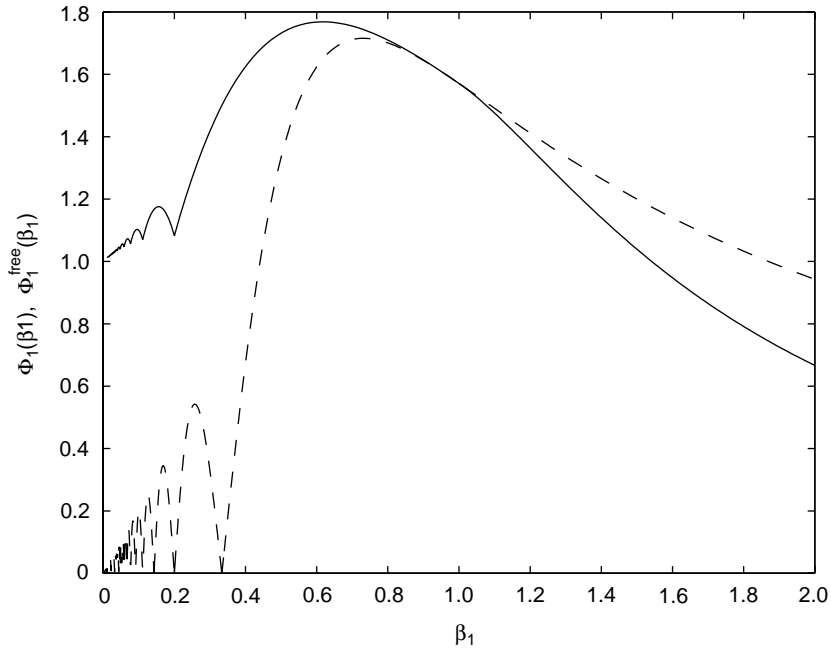
Fig. 1. Functions $\Phi_1(\beta_1)$ (——) and $\Phi_1^{free}(\beta_1)$ (- - -).

4.2.3. Maximum response

Fig. 1 explicitly shows that, if the velocity of the travelling force is less than $v_1 = \pi$ ($\beta_1 < 1$), the maximum response of the forced vibration is greater than the amplitude of the free vibration. If $v > v_1$, the maximum response occurs when the force leaves the beam. Denote

$$\Psi_1(\beta_1) = \begin{cases} \Phi_1(\beta_1) & \text{for } 0 < \beta_1 \leqslant 1, \\ \Phi_1^{free}(\beta_1) & \text{for } 1 \leqslant \beta_1 < \infty. \end{cases} \tag{14}$$

Then,

$$W_1(v) \equiv \max_x \ \max_t |w_1(x,t;v)| = \frac{2}{\omega_1^2} \Psi_1(\beta_1) \equiv \frac{2}{\pi^4} \Psi_1\left(\frac{v}{\pi}\right). \tag{15}$$

The maximum of the function $\Psi_1(\beta_1)$ is approximately equal to 1.75 (it should be noted that this value has been obtained earlier in [6]) and is attained at $\beta_1 \approx 0.62$ ($v \approx 2$). As v decreases, $\Phi_1(\beta_1)$ tends to one, which corresponds to the maximum response of the beam given by $2/\pi^4 \approx 1/48.7$, which is very close to the exact static deflection $\frac{1}{48}$ of the beam due to the unit force applied at the midspan.

4.3. Modal response functions for the higher modes

Note that the results of Section 4.2 can be applied with small changes to finding maxima of the higher terms $w_n(x,t) = \varphi_n(x)q_n(t)$ of expansion (4), $W_n(v) = \max_x \max_t |w_n(x,t;v)|$. These functions will be referred to in what follows as the *modal response functions* (MRF).

Differentiating Eq. (8), we find that the values of t at which local maxima of the $|q_n(t)|$ are attained are again given by Eq. (9) and that the functions describing these maxima are given by Eq. (10) upon substitution of β_n and ω_n for β_1 and ω_1, where

$$\beta_n = \frac{\alpha_n}{\omega_n} \equiv \frac{\lambda_n v}{\omega_n} \equiv \frac{v}{v_n}, \tag{16}$$

and $v_n = \omega_n/\lambda_n = nv_1 = n\pi$. Using the constraint $t(\beta_n, k) \leqslant T_p$, one can determine the domain of each of the functions (10) and construct the function describing the global maximum of the forced response due to the excitation of the nth mode. As n increases, the corresponding formulas become more involved for small values of β_n. However, they can be very well approximated by linear functions in this region. For example, for the second mode, we have

$$\Phi_2(\beta_2) = \begin{cases} 1 + 6(\Phi_{15}(1/6) - 1)\beta_2, & 0 \leqslant \beta_2 \leqslant 1/6, \\ \Phi_{15}(\beta_2), & 1/6 \leqslant \beta_2 \leqslant 1/5, \\ \Phi_{14}(\beta_2), & 1/5 \leqslant \beta_2 \leqslant 1/4, \\ \Phi_{11}(\beta_2), & 1/4 \leqslant \beta_2 \leqslant 1/2, \\ \Phi_{12}(\beta_2) & 1/2 \leqslant \beta_2 \leqslant 1, \end{cases} \tag{17}$$

with the relative error of the linear approximation being less than 1%.

The amplitude of free beam vibration due to the nth mode is calculated in exactly the same way as in Section 4.2 resulting in the equation

$$|C_n(\beta_n)| = \frac{2}{\omega_n^2} \Phi_n^{free}(\beta_n), \tag{18}$$

where

$$\Phi_n^{free}(\beta_n) = \frac{2\beta_n}{|\beta_n^2 - 1|} \left| \sin \omega_n \left(\frac{1}{2\pi} + \frac{T_p}{2} \right) \right| \equiv \frac{2\beta_n}{|\beta_n^2 - 1|} \left| \sin \frac{n\pi}{2} \left(\frac{n\beta_n + 1}{\beta_n} \right) \right|. \tag{19}$$

The functions $\Phi_2(\beta_2)$ and $\Phi_2^{free}(\beta_2)$ are depicted in Fig. 2 by the solid and dashed lines, respectively. As can be seen, the amplitude of the free vibration is less than the maximum forced response for $\beta_2 \leqslant 1$ and exceeds it if $\beta_2 > 1$ ($v > 2\pi$). Thus, the MRF for the second mode is given by

$$W_2(v) \equiv \max_x \ \max_t w_2(x, t; v)| = \frac{2}{\omega_2^2} \Psi_2(\beta_2) \equiv \frac{1}{8\pi^4} \Psi_2\left(\frac{v}{2\pi} \right), \tag{20}$$

where

$$\Psi_2(\beta_2) = \begin{cases} \Phi_2(\beta_2), & \text{for } 0 < \beta_2 \leqslant 1, \\ \Phi_2^{free}(\beta_2) & \text{for } 1 \leqslant \beta_2 < \infty. \end{cases} \tag{21}$$

The maximum value of $\Psi_2(\beta_2)$ over all velocities is attained at $\beta_2 \approx 0.85$ and is approximately equal to 3.25. This function is not specific to any particular beam and describes the dependence on the dimensionless velocity of the maximum response of the beam due to the excitation of the second mode only.

Function $\Psi_2(\beta_2)$ given by Eq. (21) can be used to calculate the maximum response of a beam with an additional support at the middle point. Indeed, the first eigenfrequency and
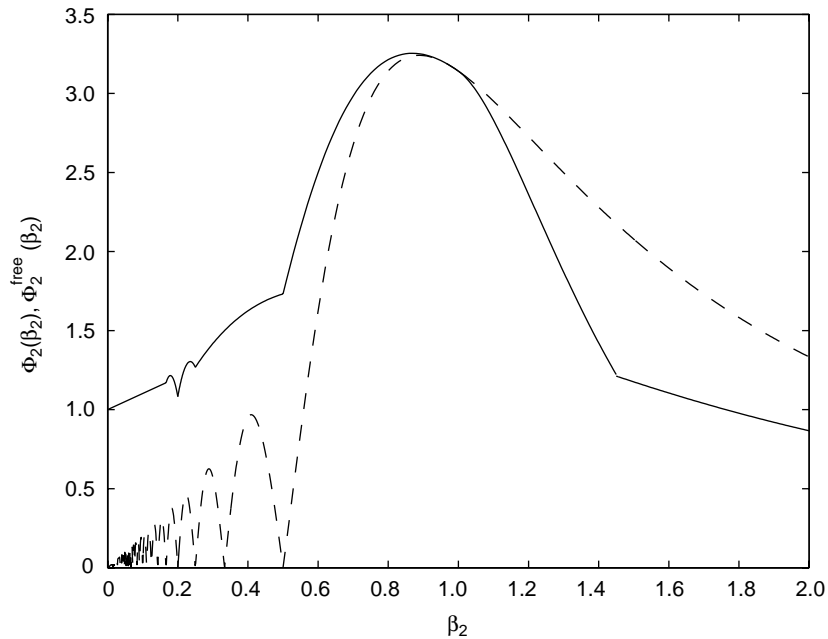
Fig. 2. Functions $\Phi_2(\beta_2)$ (——) and $\Phi_2^{free}(\beta_2)$ (- - -).

eigenfunction of such a beam coincide with the second eigenfrequency and eigenfunction of an SS beam without the additional support. Then, the maximum response of this beam is given by Eq. (20).

Comparison of $\Psi_2(\beta_1)$ and $\Psi_2(\beta_2)$, with regard to Eqs. (15), (20) and the relation $\beta_1 = 2\beta_2$, shows that, if $v < v_1$, $W_2(v)$ is less than $W_1(v)$ by the factor of $\omega_2^2/\omega_1^2 = 16$. The ratio $W_2(v)/W_1(v)$ is maximal in the neighborhood of the point $\beta_2 = 0.85$, i.e., when $v \approx 0.85v_2 = 1.7v_1$ and is approximately equal to $\frac{1}{5}$. When $\beta_1 \to \infty$ $(v \gg \pi)$, the ratio can be shown to have the asymptotic $(\omega_1^2/\omega_2^2)8\pi/\beta_1^2 = \pi^3/2v^2$. To get an idea of relative magnitudes of the modal response functions $W_1(v)$, $W_2(v)$, and $W_3(v)$, they are depicted together in Fig. 3.

It is well-known (and can easily be shown by using the technique described above) that, for small velocities, the contribution of the higher modes can be estimated as $\omega_n^2/\omega_1^2 = (1/n)^4$ and thus can certainly be neglected. Therefore, we only need to estimate the effect of the nth mode in the neighborhood of the velocity where $\max_t|q_n(t)|$ takes its maximum value and to examine the asymptotics for large values of the velocity. It can be shown that the maximum of the function $W_n(v)$ is attained at $\beta_n \approx 1$ $(v \approx n\pi)$ and is given by

$$\frac{2}{\omega_n^2} \max_{\beta_n} \Phi_{1n}(\beta_n) \approx \frac{2}{(n\pi)^4} \frac{n\pi}{2} = \frac{1}{\pi^3 n^3}.$$

Then, it follows that, for $v \approx n\pi$, $W_n(v)/W_1(v) \approx \pi/4n^2$. The asymptotics of the function $W_n(v)/W_1(v)$ for large v, $v \gg v_n$, can be shown to be $(1/n)^3$.
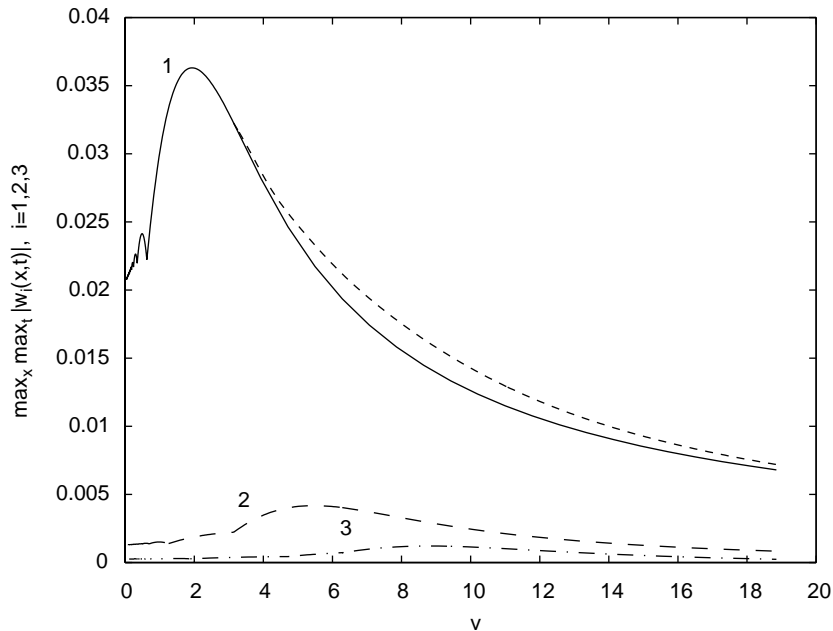
Fig. 3. Modal response functions $W_1(v)$ (curve 1), $W_2(v)$ (curve 2), $W_3(v)$ (curve 3) of an SS beam and the "improved" function (28) (- - -).

4.4. Contribution of the higher modes to the maximum response

The modal response function $W_n(v)$ shows the maximum value of the nth term of expansion (4) and gives an idea of the velocity range where the effect of the nth term is maximal. However, the maximum beam response $W(v)$ is not an algebraic sum of the MRFs. Since the maxima of different terms of expansion (4) are attained at different t and x, the contribution of the higher terms to the maximum response is, in fact, considerably less than it may seem from the comparison of the modal response functions. To demonstrate this and to see how the second and third modes contribute to the maximum beam response, we calculate the three-mode approximation of the maximum beam response. Since, as was shown above, the maxima of $|q_2(t)|$ and $|q_3(t)|$ are much less than that of $|q_1(t)|$, we can treat the second and third terms of expansion (4) as perturbations and assume that the maximum of $|w(x, t; v)|$ is attained at $x = 0.5 + \Delta x$ and $t = t_0 + \Delta t$, where $t_0(v) = \arg \max_t |q_1(t; v)|$ is given by one of functions (9) and Δx and Δt are small.

Let first, $v \leqslant v_1 = \pi$. The three-mode approximation of the response is given by

$$w(x, t; v) = \sqrt{2}\{q_1(t)\sin \pi x + q_2(t)\sin 2\pi x + q_3(t)\sin 3\pi x\}$$
$$= \sqrt{2}q_1(t_0)\{\tilde{q}_1(t)\sin \pi x + \varepsilon(v)\tilde{q}_2(t)\sin 2\pi x + \delta(v)\tilde{q}_3(t)\sin 3\pi x\},$$

where $\varepsilon(v) = q_2(t_0)/q_1(t_0)$ and $\delta(v) = q_3(t_0)/q_1(t_0)$ are small parameters (it follows from the above discussions that $|\varepsilon(v)| \leqslant 1/16$ and $|\delta(v)| \leqslant 1/81$), $\tilde{q}_1(t)$, $\tilde{q}_2(t)$, and $\tilde{q}_3(t)$ are of the same order of

magnitude, and $\tilde{q}_1(t_0) = \tilde{q}_2(t_0) = \tilde{q}_3(t_0) = 1$. Then, expanding all functions into Taylor series in the neighborhood of $x = 0.5$ and $t = t_0$ and ignoring small terms of order greater than two, we find that $\Delta w \equiv w(0.5 + \Delta x, t_0 + \Delta t; v)$ is given by

$$\Delta w \approx \sqrt{2} q_1(t_0) \left\{ 1 - \ddot{\tilde{q}}_3(t_0) \frac{(\Delta t)^2}{6} - \frac{(\pi \Delta x)^2}{2} - 2\varepsilon(\pi \Delta x) - \delta - \delta \dot{\tilde{q}}_3(t_0) \Delta t \right\}. \tag{22}$$

Equating the derivatives with respect to Δt and $(\pi \Delta x)$ to zero, we obtain

$$-\pi \Delta x - 2\varepsilon = 0, \quad -\ddot{\tilde{q}}_1(t_0)\Delta t - \delta \dot{\tilde{q}}_3(t_0) = 0.$$

Then, it follows that the maximum of Δw is attained for

$$\Delta x = -\frac{2\varepsilon}{\pi} \tag{23}$$

and $\Delta t = O(\delta)$. The particular dependence of Δt on δ is of no importance since the right-hand of Eq. (22) linearly depends on δ, whereas the terms containing Δt have the higher order of smallness and can be ignored. The corresponding value of the maximum response is then given by

$$W(v) = \max_{\Delta x} \max_{\Delta t} |\Delta w| = W_1(v) \left\{ 1 - \delta + 2\varepsilon^2 + o(\delta) + o(\varepsilon^2) \right\}, \quad 0 \leqslant v \leqslant \pi. \tag{24}$$

Let now, $v > v_1$. In this case, the maximum response occurs when the force leaves the beam and the function $W_1(v)$ is equal to the amplitude $|C_1(v)|$ of the free vibration at the first eigenfrequency. Since, as was shown, $|C_1(v)|$ is much greater than the amplitudes of the other eigenvibrations, we can again calculate an approximate amplitude of the resulting vibration by using perturbation theory. It follows from Eq. (A.3), derived in Appendix A, that the amplitude of the free vibration can be calculated as

$$|C(v)| = \max_x (C_1(v)\sin \pi x \pm C_2(v)\sin 2\pi x + C_3(v)\sin 3\pi x), \tag{25}$$

where $C_n(v), n = 1, 2, 3$, is the amplitude with sign of the nth beam eigenvibration, given by Eq. (A.2). Similar to the case of the forced vibration, we introduce the small parameters $\varepsilon(v) = C_2(v)/C_1(v)$ and $\delta(v) = C_3(v)/C_1(v)$ and rewrite Eq. (25) in the form

$$|C(v)| = |C_1(v)| \max_x (\sin \pi x \pm \varepsilon(v)\sin 2\pi x + \delta(v)\sin 3\pi x).$$

Expanding the sines into Taylor series in the neighborhood of $x = 0.5$, we get

$$|C(v)| \approx |C_1(v)| \left\{ 1 - \frac{(\pi \Delta x)^2}{2} \pm \varepsilon \pi \Delta x - \delta + \delta \frac{9(\pi \Delta x)^2}{2} \right\}. \tag{26}$$

The last term in this equation has the higher order of smallness compared to the other three terms; however, we do not ignore it in order to improve the approximation (in addition, it represents the combined effect of the second and third modes on the response). Differentiating the right-hand of Eq. (26) with respect to Δx, we obtain

$$\Delta x = \pm \frac{2\varepsilon}{1 - 9\delta}. \tag{27}$$

Substituting the Δx into Eq. (26), we finally obtain

$$W(v) = |C(v)| \approx |C_1(v)| \left\{ 1 - \delta + \frac{2\varepsilon^2}{1 - 9\delta} \right\}, \quad v \geqslant v_1. \tag{28}$$

The right-hand of Eq. (28) is shown in Fig. 3 by the dashed line. Note that the improved solution (24) for the forced response is not depicted since it coincides with the unperturbed function $W_1(v)$.

Combining both cases considered, we get

$$W(v) = W_1(v)(1 + r(v)),$$

where the function

$$r(v) = \begin{cases} -\delta(v) + 2\varepsilon^2(v), & 0 \leqslant v \leqslant \pi, \\ -\delta(v) + \dfrac{2\varepsilon^2(v)}{1 + 9\delta(v)}, & v \geqslant \pi, \end{cases} \tag{29}$$

used to improve the one-mode approximation is depicted in Fig. 4. It explicitly shows that the accuracy of the one-mode approximation of the maximum response is very good in a wide range of the velocities. In the most interesting range of velocities, $0 < v < 4$ (where the maximum response is greater then the static deflection), the relative error is less than 2%. This result substantiates the assumption that the one-mode approximation of the maximum response is quite adequate.
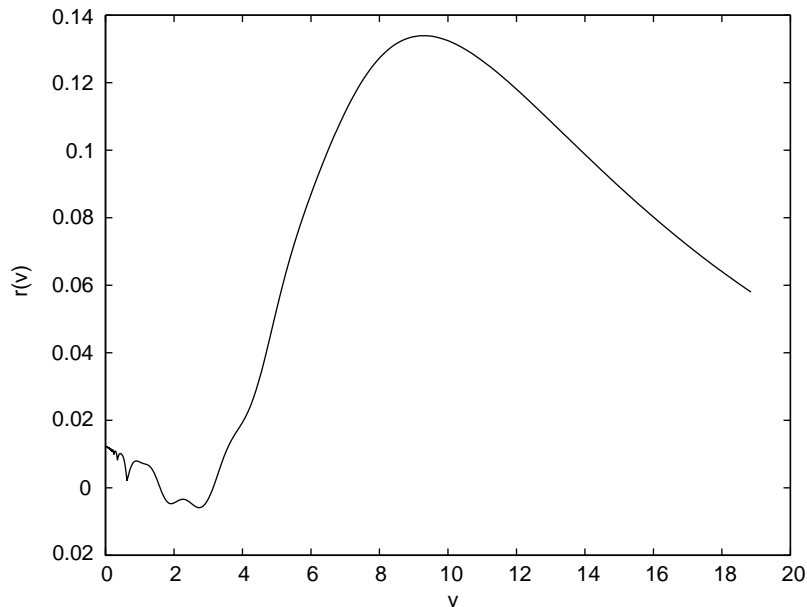

Fig. 4. The relative error (29) of the one-mode approximation for an SS beam.

5. CC Beam

In the case of a CC beam, the analytical construction of the function $W(v)$ seems to be too complicated, and we constructed it numerically. In our experiments, letting v take discrete values v_i, $i = 1, 2, \ldots$, we numerically calculated the time-dependent coefficients $q_1(t, v_i)$ and found their maxima, $\max_t |q_1(t, v_i)|$. Using the values of the time-dependent coefficients and their derivatives at $t = T_p$, we calculated the amplitude of free vibration $C_{1cc}(v_i)$. As in the case of an SS beam, a velocity v_{1cc} exists such that, for $v < v_{1cc}$, the maximum response occurs at a time when the force is on the beam and, for $v > v_{1cc}$, the maximum deflection takes place after the force leaves the beam (free vibration). The value of v_{1cc} was experimentally found to be about twice the critical velocity v_1 for an SS beam, $v_{1cc} \approx 2\pi$. The plots of the functions

$$\Phi_{1cc}(v) = \frac{\omega_{1cc}^2}{\varphi_{1cc}(0.5)} \max_t |q_1(t; v)|, \qquad \Phi_{1cc}^{free}(v) = \frac{\omega_{1cc}^2}{\varphi_{1cc}(0.5)} C_{1cc}(v),$$

(analogues of $\Phi_1(\beta_1)$ and $\Phi_1^{free}(\beta_1)$ for an SS beam shown in Fig. 1) are depicted in Fig. 5 versus the velocity v/v_{1cc}. As can be seen from comparison of Figs. 1 and 5, the curves describing the maximum forced response and the amplitude of the free vibration for SS and CC beams are similar enough to each other such that one can use the functions $\Phi_1(\beta_1)$ and $\Phi_1^{free}(\beta_1)$, which are found in analytical form, instead of the functions $\Phi_{1cc}(\beta_1)$ and $\Phi_{1cc}^{free}(\beta_1)$ upon substitution of $v/(2\pi)$ for β_1.
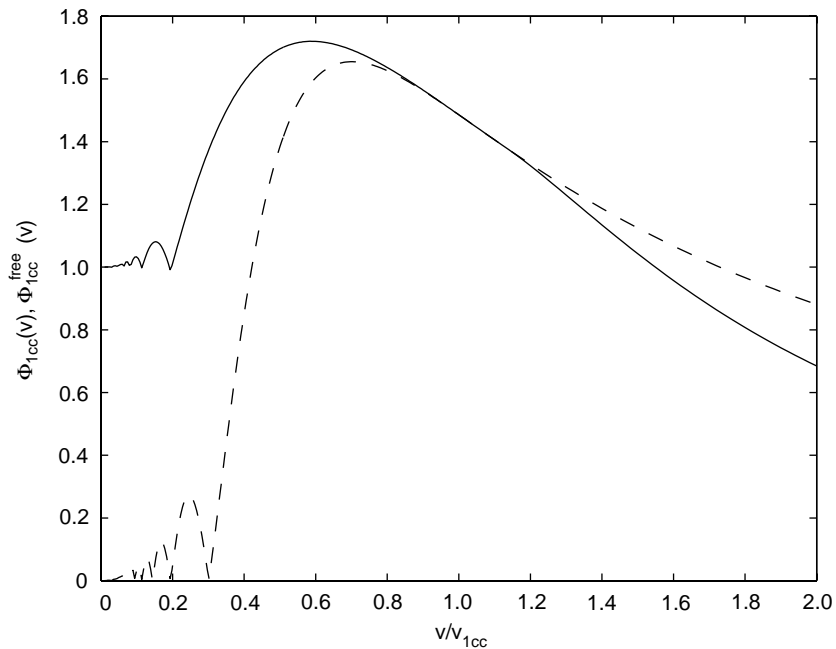

Fig. 5. Functions $\Phi_{1cc}(v/2\pi)$ (——) and $\Phi_{1cc}^{free}(v/2\pi)$ (- - -).

The maximum response of the CC beam is calculated as

$$W_{1cc}(v) = \frac{\varphi_{1cc}^2(0.5)}{\omega_{1cc}^2} \max\left\{\Phi_{1cc}(v), \Phi_{1cc}^{free}(v)\right\}.$$

The plot of the function $W_{1cc}(v)$ obtained is shown in Fig. 6 (curve 1). This function describes the one-mode approximation of the maximum response of the beam as a function of the dimensionless velocity. Curves 2 and 3 in this figure show the functions $W_{2cc}(v)$ and $W_{3cc}(v)$, respectively. These functions were also calculated numerically in a similar way by using the functions $q_2(t)$ and $q_3(t)$. The maximum response of a CC beam over all velocities occurs at $v \approx 3.6$, which is about 1.8 times greater than that of an SS beam. The value of the maximum response of a CC beam is about 25% of that of the same beam with simply supported ends, which is approximately equal to $(\varphi_{1cc}(0.5)/\varphi_{1ss}(0.5))^2(\omega_{1ss}/\omega_{1cc})^2$, where the superscripts *ss* and *cc* are used to denote the eigenfrequencies and eigenfunctions of the SS and CC beams, respectively.

It can be seen from Fig. 6 that the ratios $W_{2cc}(v)/W_{1cc}(v)$ and $W_{3cc}(v)/W_{1cc}(v)$ are greater than those for an SS beam, which can be explained by the fact the ratios of the second and third eigenfrequencies to the first one are less than those for an SS beam. The relative error of the one-mode approximation of the maximum response can be calculated similar to the case of an SS beam (Section 4.4). For $v < 2\pi$, the relative error was shown to be less than 3%.
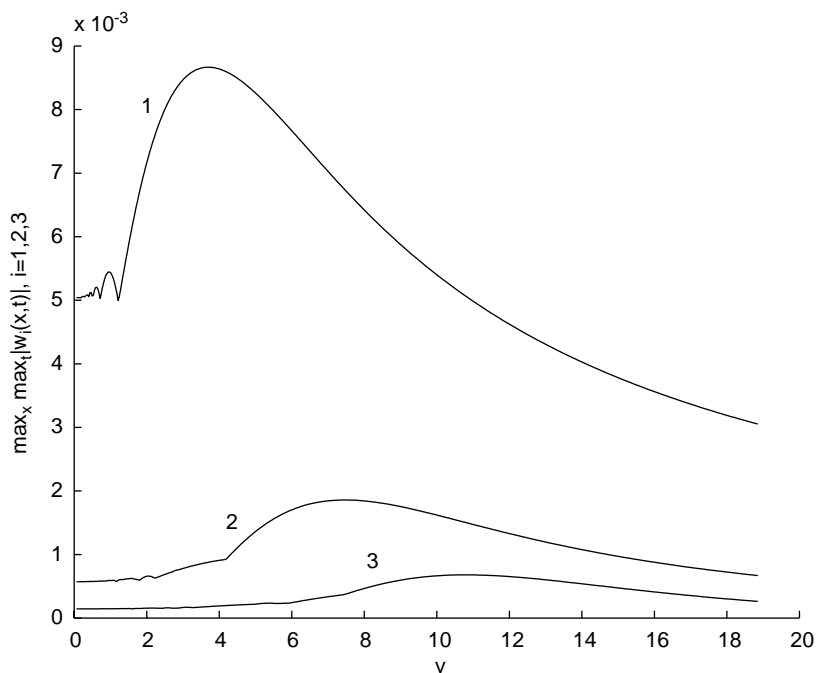

Fig. 6. MRFs $W_1(v)$ (curve 1), $W_2(v)$ (curve 2), and $W_3(v)$ (curve 3) of a CC beam.

6. Discussion of the results obtained

As can be seen from the above discussion, the use of the one-mode approximation for the calculation of the maximum response is quite adequate for engineering applications. The functions $W_1(v)$ and $W_{1cc}(v)$ (Figs. 3 and 6) constructed in Sections 4 and 5 make it possible to calculate the maximum response of an arbitrary SS or CC beam by simply rescaling the plots. A summary of some analytical results (simple, ready-to-use, rules) is given below. Note that these results are formulated in terms of the first beam eigenfrequency ω_1 and the critical velocity $v_1 = \omega_{1ss}L/\pi$ in order to make them applicable to both the dimensional and dimensionless beams. The value of the maximum response for a dimensional beam is obtained (see Eq. (3)) from that for the dimensionless beam by multiplying by the factor FL^3/EI.

(1) The maximum response of an SS beam over all velocities occurs when the force is on the beam at $v = v_{peak} \approx 0.62v_1$ and is equal to $W_1(v_{peak}) \approx 3.5/\omega_1^2$, which is about 80% more than the static deflection of the beam due to the force of the same magnitude applied at the midspan of the beam. If the passage time is less than the half-period of the first beam eigenvibration $(v > v_1)$ the maximum response occurs when the force leaves the beam. If $v > 2v_1$, the maximum response is less than the static deflection (Fig. 1).

(2) The maximum response of a CC beam over all velocities is attained at $v \approx 1.8v_{peak}$ and is approximately equal to $1.7\varphi_1^2/\omega_1^2 \approx 4.3/\omega_1^2$, which is about 4.2 times less than that of the same beam with simply supported ends. If $v > 2v_1$, the maximum response of a CC beam takes place when the force leaves the beam (Fig. 5).

(3) In the neighborhood of maximum response over all velocities, variation of the velocity by 25% results in variation of the maximum response of the order of 5% (Figs. 1 and 5).

(4) To reduce the maximum response by about 20%, the velocity must be either half of v_{peak} or twice as much as v_{peak} (Fig. 1).

(5) The amplitude of the free vibration of an SS beam is minimal (the first beam eigenvibration is completely suppressed) when $\beta_1 = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$; these values of β_1 correspond to the velocities for which the passage time T_p is an odd multiple of the half-period of the first beam eigenvibration: $T_p = 3(T_1/2), 5(T_1/2), 7(T_p/2), \ldots$ (Fig. 1).

(6) If an SS beam has an additional support at the middle point, the maximum response at $v = v_{peak}$ is about 20 times less than that of the beam without the support. The maximum response over all velocities of such a beam is approximately 8.5 times less than $W_1(v_{peak})$ and is attained at $v \approx 2.7v_{peak}$ (Figs. 1 and 2).

(7) For an SS beam, the one-mode approximation $W_1(v)$ gives the exact solution for the maximum response at $v = v_1$.

7. Conclusions

The problem of finding the maximum response of a beam subject to a travelling force has been examined. It has been shown that there exists a unique function describing the dependence of the maximum response of an arbitrary beam with given boundary conditions on the velocity of the travelling force. This function allows for convenient assessment of the maximum response for

the MFP without intensive computations. This result can be quite useful for design engineers and researchers facing the MFP.

The technique to find an approximate solution to the problem under examination based on the use of one term of the expansion of the solution has been suggested. The maximum response as a function of the velocity of the travelling force has been calculated in explicit form for a simply supported (SS) beam and constructed numerically for a clamped–clamped (CC) beam. The relative error of the one-mode approximation of the maximum response in the neighborhood of the "dangerous" velocity (where the maximum response takes its maximum value) has been shown to be less than 1% for an SS beam and 3% for a CC beam, which is quite sufficient for engineering applications. The basic advantage of the results reported herein is that they provide the engineer with very simple tools making it possible to get answers to a number of important questions without solving the moving force problem. While only the SS and CC beams have been considered, the response–velocity dependence function examined in this study is generic for structural systems, and the method used in this work can be applied to finding the maximum response functions for other structural systems.

Acknowledgements

Appendix A. Free vibration of the beam after the force leaves it

After the force leaves the beam, the equation governing its vibration is well-known to be

$$w(x, t) = \sum_{n=1}^{\infty} \left\{ A_n \cos \omega_n(t - T_p) + B_n \sin \omega_n(t - T_p) \right\} \sqrt{2} \sin n\pi x, \quad t \geqslant T_p, \tag{A.1}$$

where the coefficients A_n and B_n are easily obtained by comparing Eqs. (4) and (A.1) at $t = T_p$ [5] with regard to the orthogonality of the eigenfunctions: $A_n = q_n(T_p)$ and $B_n = \dot{q}_n(T_p)/\omega_n$. Using Eq. (8) and the equality $\alpha_n T_p = n\pi$ and denoting $\gamma_n = \sqrt{2}\alpha_n/\omega_n(\alpha_n^2 - \omega_n^2)$, we find that

$$q_n(T_p) = \gamma_n \sin \omega_n T_p, \qquad \dot{q}_n(T_p) = \gamma_n \omega_n(\cos \omega_n T_p - (-1)^n).$$

Then we have

$$A_n \cos \omega_n(t - T_p) + B_n \sin \omega_n(t - T_p)$$

$$= q_n(T_p)\cos \omega_n(t - T_p) + \frac{\dot{q}_n(T_p)}{\omega_n} \sin \omega_n(t - T_p)$$

$$= \gamma_n \left\{ \sin \omega_n T_p \cos \omega_n(t - T_p) + \cos \omega_n T_p \sin \omega_n(t - T_p) - (-1)^n \sin \omega_n(t - T_p) \right\}$$

$$= \gamma_n \left\{ \sin \omega_n t - \sin \omega_n \left(t - \frac{1}{\pi} - T_p \right) \right\} = 2\gamma_n \sin \omega_n \left(\frac{1}{2\pi} + \frac{T_p}{2} \right) \cos \omega_n \left(t - \frac{1}{2\pi} - \frac{T_p}{2} \right).$$

Substituting this into Eq. (30) and denoting

$$C_n(v) = \frac{4\alpha_n}{\omega_n(\alpha_n^2 - \omega_n^2)} \sin \omega_n \left(\frac{1}{2\pi} + \frac{T_p}{2} \right), \tag{A.2}$$

we get

$$w(x,t) = \sum_{n=1}^{\infty} C_n(v) \cos \omega_n \left(t - \frac{1}{2\pi} - \frac{T_p}{2} \right) \sin n\pi x, \quad t \geqslant T_p. \tag{A.3}$$

Then it follows that the free vibration is the sum of the eigenvibrations at the eigenfrequencies ω_n with the amplitudes $|C_n(v)|$ and is periodic with the period being equal to the period of the first eigenvibration $T_1 = 2/\pi$. It can be checked directly that, when $\cos \omega_1(t - 1/2\pi - T_p/2) = \pm 1$, $\cos \omega_n(t - 1/2\pi - T_p/2)) = (\pm 1)^n$. This implies that all modes of the vibration at these times are either in phase or out of phase, depending on the signs of the factors $C_n(v)$.

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